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# The work of Professor Jun-iti Nagata

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ABSTRACT

This is a survey article on the work of Professor Jun-iti Nagata. We present his educational and professional career, his activities and contributions to the topology community and his work on topology, especially the theory of uniform spaces, rings of continuous functions, metrization and generalized metric spaces.



## 1. Professor Jun-iti Nagata

Professor Jun-iti Nagata, a honorary professor at Osaka Kyoiku University, passed away on November 6, 2007. For over a half century, he was one of the world leaders in general topology. We are saddened that the international community of topologists, especially the general topologists, have lost a distinguished leader.

By recalling from [HY] we present his educational and professional career. Jun-iti Nagata was born on March 4, 1925 in Osaka. He graduated from Naniwa High School in 1944, and Tokyo Imperial University (now, the University of Tokyo) in 1947, directed by Professor S. Iyanaga. He was awarded the Doctor of Science degree in 1956 from Osaka University. His career as a researcher began with an appointment as a research assistant in the Department of Mathematics of Osaka University in 1948. One year later, he moved to Osaka City University and became a lecturer when Osaka City University was founded. Later, he was an associate professor (1955–1961) and a professor (1961–1965) at Osaka City University. His first professorship abroad was as a Visiting Professor at the University of Washington in 1959. He also served as a member of the Institute for Advanced Study in 1963–1964. During his stay at the Institute for Advanced Study, he wrote the book *Modern General Topology*. After visiting the Institute for Advanced Study, he was a professor at the University of Pittsburgh (1965–1975). Then, he was a professor at the University of Amsterdam (1975–1982) as a successor of Professor J. de Groot. He came back to Japan in 1982 as a professor at Osaka Kyoiku University (1982–1990). After his retirement from Osaka Kyoiku University, he was a professor at Osaka Electro-Communication University (1990–1995). As mentioned above, Nagata was a professor at several universities in the world: Japan, United States and the Netherlands.

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Nagata has made essential contributions to the development of topology, especially, the theory of metrization, generalized metric spaces, dimension theory and the theory of rings of continuous functions. All topologists know the Nagata–Smirnov Metrization Theorem: *A regular space X is metrizable if and only if X has a $\sigma$-locally finite base*. The theories of dimension and metrization are the major subjects of his work. We can see in his book *Modern Dimension Theory* (1965), which was extended and revised in 1983, both theories were linked with each other by him.

The details of his effort in the theory of metrization, generalized metric spaces, and rings of continuous functions will be presented in the following sections. His research in dimension theory appears in a separate article of Hodel [Ho3] in this issue. The reader should also refer to [A3,H,MP,M,Pa] for the summaries of Nagata's work.

Nagata was the author of 96 articles in the period from 1949 to 2007. Nagata proved many classic theorems, and we can find some of them in the standard topology textbooks such as [E1,E2,Na2,Pe]. As we mentioned above, he published two books on topology: Modern Dimension Theory (1965) [38] (revised in 1983 [67]) and Modern General Topology (1968) [41] (revised in 1985 [57]). Both books are standard textbooks for researchers and graduate students who study topology. His book on dimension theory [38] was published 25 years after the well-known book of Hurewicz and Wallman [HW], which is concerned with the dimension of separable metrizable spaces. Nagata extended the progress in dimension theory from [HW] by adding the dimension theory in general metric spaces. We can easily surmise that he was motivated by the development of the dimension theory in general metrizable spaces (e.g., the Katětov–Morita Theorem on the coincidence of the covering dimension and the large inductive dimension) and the metrization theorems (e.g., Nagata–Smirnov–Bing Metrization Theorem). As S. Mardešić predicted in the review of the book, the book did become a new standard reference in dimension theory instead of [HW].

He also edited the monograph; *Topics in General Topology* (1989) with K. Morita, and the encyclopedia; *Encyclopedia of General Topology* (2004) with K.P. Hart and J. Vaughan.

Nagata was interested in all activities which contributed to the topology community. He frequently attended international conferences and symposiums, and also organized conferences, symposiums and seminars. In particular, he attended the Prague Topological Symposium several times, which is one of the most important conferences in the international community of topology. We know by his remarks in [96] that he never forgot the first Prague Symposium: "The first Prague Topological Symposium was especially impressive, because the leaders of the old (first half of 20th century) general topology (P.S. Alexandroff, K. Borsuk, M. Fréchet, K. Kuratowski, M.H. Stone) met the leaders of the new (second half of 20th century) general topology (R.D. Anderson, A.V. Arhangel'skii, R.H. Bing, C.H. Dowker, E. Eilenberg, R. Engelking, Z. Frolik, J. de Groot, E. Hewitt, M. Katětov, E. Michael, B.A. Pasynkov and Yu.M. Smirnov)." (Of course, Nagata was one of the leaders of new general topology even though he did not list himself.)

In 1970, Nagata went to a lot of work to organize and hold the International Pittsburgh Conference on General Topology in Pittsburgh, Pennsylvania, USA, under the sponsorship of the University of Pittsburgh. The Proceedings of this Conference was the start of the journal *General Topology and its Applications*, which was published continuously until now, changing into *Topology and its Applications*. Nagata was a charter organizer of the Soviet–Japan Joint Symposium since 1986. The first Soviet–Japan Joint Symposium was held at Tokyo in June, 1986, and then the symposium was held at Khabarovsk in September, 1989, Niigata in 1992 and Moscow in 1995. Unfortunately, the symposium was held only four times. Nagata also contributed to the topology community by editing several journals. He was a member of the editorial board of the following international journals: *Questions and Answers in General Topology* (Managing Editor), *Topology and its Applications* (Editor and Advisor), *Houston Journal of Mathematics* (Editor), *Scientiae Mathematicae Japonicae* (Editor, Japan), and *Rendiconti del Circolo Matematico di Palermo* (Editor, Italy).

It should be noted that Nagata wrote several survey papers and essays, and he asked several open questions in these survey articles. It was important for him that he pose open questions which could be interesting and give some direction for further research. For this purpose, he founded a journal, *Questions and Answers in General Topology* in 1980, and he managed it for over 25 years.

To this day his work in topology still stands in the center of general topology, and the questions posed by him still throw a guiding light on the future research in general topology.

## 2. Uniform spaces, rings of continuous functions and others

He began his research by the study on uniform spaces and rings of continuous functions. We guess that he might have been influenced by the monographs of A. Weil, *Sur les espaces à structure uniform et sur la topologie générale* [W] and J.W. Tukey: *Convergence and uniformity in topology* [T], because he said in the reminiscences [86] that he read the books when he was a university student. The first journal paper [1] of Nagata is concerning the rings of continuous functions on uniform spaces, and in the paper, he characterized the topologies of Tychonoff spaces by use of the rings of continuous functions. Let $X$ be a Tychonoff space. We denote by $C_p(X)$ the space of all real valued continuous functions on $X$ with the topology of pointwise convergence. In the paper, he proved

**Theorem 2.1.** *([1]) For Tychonoff spaces $X$ and $Y$ if $C_p(X)$ and $C_p(Y)$ are topologically isomorphic, then the spaces $X$ and $Y$ are homeomorphic.*

This theme is one of the major parts of his research, and the paper influenced further studies in the theory of the rings of continuous functions and topological function spaces (cf. [A2,M]). In the same paper, he also gave a characterization of complete metric spaces and totally bounded uniform spaces by some families of real-valued functions.

Nagata extended the above result to generalized metric spaces in [61] by use of the infinite operations in the ring $C^*(X)$ of bounded continuous real-valued functions of a Tychonoff space $X$. (Many of the results obtained in the paper can be extended to the ring $C(X)$ of real-valued continuous functions of $X$ with no or slight modifications.) Let $X$ be a Tychonoff space and $\{f_\alpha\colon \alpha \in A\}$ a subset of $C^*(X)$. Let $\bigwedge_\alpha f_\alpha(x) = \inf\{f_\alpha(x)\colon \alpha \in A\}$, and $\bigvee_\alpha f_\alpha(x) = \sup\{f_\alpha(x)\colon \alpha \in A\}$. A subset $L_0$ of $C^*(X)$ is called *normal* if $\bigwedge_\alpha f_\alpha \in L_0$ and $\bigvee_\alpha f_\alpha \in L_0$ for every subset $\{f_\alpha\colon \alpha \in A\}$ of $L_0$. A sequence $L_1, L_2, \ldots$ of normal subsets of $C^*(X)$ is called a normal sequence. A subset $L$ of $C^*(X)$ is said to be *$\sigma$-normally generated* by the normal sequence $\{L_i\colon i = 1, 2, \ldots\}$ if $L = \{f \in C^*(X)\colon$ for every $\varepsilon > 0$ there are subsets $\{f_\beta\colon \beta \in B\}$ and $\{f_\gamma\colon \gamma \in C\}$ of $\bigcup_{i=1}^\infty L_i$ such that $\|\bigwedge_\beta f_\beta - f\| < \varepsilon$ and $\|\bigvee_\gamma f_\gamma - f\| < \varepsilon\}$. Nagata showed that a Tychonoff space $X$ is metrizable if and only if $C^*(X)$ is $\sigma$-normally generated by the normal sequence. Furthermore, he obtained a characterization of paracompact $M$-spaces in terms of $C^*(X)$. The $M$-spaces were introduced by K. Morita [Mo3] and are equivalent to $p$-spaces introduced by A.V. Arhangel'skii [A1] for paracompact spaces. (The definition and more details on paracompact $M$-spaces will appear in the next section.) A maximal ideal $J$ in $C^*(X)$ is called *free* if there is a subset $\{f_\alpha\colon \alpha \in A\}$ of $J$ such that $\bigvee_\alpha f_\alpha \in C^*(X) - J$. A maximal ideal is called *fixed* if it is not free. A subset $K$ of $C^*(X)$ is called *free* if $K \not\subset J$ for every fixed maximal ideal $J$.

**Theorem 2.2.** *([61]) A Tychonoff space $X$ is a paracompact $M$-space if and only if there is a $\sigma$-normally generated subring $L$ of $C^*(X)$ such that for every free maximal ideal $J$ in $C^*(X)$, $J \cap L$ is free.*

Similar characterizations in terms of $C^*(X)$ for paracompact Čech complete spaces and $G_\delta$-spaces are also given in the same paper, where a Tychonoff space is called a *$G_\delta$-space* if it is homeomorphic to a $G_\delta$-set in the product of a metric space and a compact $T_2$-space.

In [11], Nagata refined the results obtained in the paper [1] to a general complete uniform space, and characterized such a space by a directed family of uniformly continuous functions from the subspace in the Tychonoff cube $\mathbb{I}^\tau$.

He also extended the results obtained in [1] to uniform spaces in [14] and [16]. Let $X$ be a uniform space with a uniformity $\mathbb{U} = \{\mathcal{U}_\alpha\colon \alpha \in A\}$. Although Nagata considered the general function space $F(X)$ consisting of functions defined on subsets of $X$ having the values in another uniform space $Y$, we restrict ourselves to the space $C_u(X)$ of uniformly continuous real-valued functions of $X$. Nagata gave a new uniformity $\{\mathcal{U}_{\alpha\varepsilon}\colon \alpha \in A,\ \varepsilon > 0\}$ on $C_u(X)$ as follows: $\mathcal{U}_{\alpha\varepsilon} = \{U_{\alpha\varepsilon}(f)\colon f \in C_u(X)\}$ and $U_{\alpha\varepsilon}(f) = \{g \in C_u(X)\colon$ for each $x \in X$ there are $y, y' \in U_\alpha(x)$ such that $|f(x) - g(y)| < \varepsilon$ and $|g(x) - f(y')| < \varepsilon\}$ for each $\alpha \in A$, $f \in C_u(X)$ and $\varepsilon > 0$. Then he proved the following.

**Theorem 2.3.** *([16]) Let $X$ and $Y$ be complete uniform spaces, and $C_u(X)$, $C_u(Y)$ the rings of bounded uniformly continuous real-valued functions of $X$, $Y$ respectively with the uniformities defined above. Then $X$ and $Y$ are uniformly homeomorphic if and only if $C_u(X)$ and $C_u(Y)$ are uniformly isomorphic, where a mapping is a uniformly isomorphic if it is a uniformly homeomorphism which preserves the lattice-order or the ring operation.*

He also gave characterizations of uniform topologies by several ways, e.g., the lattices of the uniformities and a uniform basis which has special properties (cf. [3,7,8,10,11,12,13]). We present here some of them. Let $X$ be a uniform space. We consider the lattices of the uniformities and a uniform basis which has the following properties:

(a) Let $\mathbb{U} = \{\mathcal{U}_\alpha\colon \alpha \in A\}$ be a uniformity of $X$. For each $\mathcal{U}_\alpha, \mathcal{U}_\beta \in \mathbb{U}$ we mean by $\mathcal{U}_\alpha < \mathcal{U}_\beta$, that $\mathcal{U}_\alpha$ is a usual refinement of $\mathcal{U}_\beta$ and by $\mathcal{U}_\alpha \lhd \mathcal{U}_\beta$, $\mathcal{U}_\alpha$ is a delta-refinement of $\mathcal{U}_\beta$. If $\mathcal{U}_\alpha < \mathcal{U}_\beta$ and $\mathcal{U}_\beta < \mathcal{U}_\alpha$ holds, then we regard $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ as the same covering. Then $\mathbb{U}$ is a lattice with respect to the order $<$.
(b) A family $\mathcal{L}$ of open uniform coverings of $X$ is called a *uniform basis* if for every open uniform covering $\mathcal{U}$ of $X$ there is $\mathcal{V} \in \mathcal{L}$ such that $\mathcal{V}$ refines $\mathcal{U}$. Let $\mathcal{L}(X)$ be a lattice of a uniform basis of $X$ which satisfies the following condition:
   (1) For each $\mathcal{U} \in \mathcal{L}(X)$ and each open set $U_0$ of $X$ there exists $\mathcal{W}(U_0, \mathcal{U}) \in \mathcal{L}(X)$ such that (i) $W \not\supset U_0$ for each $W \in \mathcal{W}(U_0, \mathcal{U})$, and (ii) $U \in \mathcal{U}$ and $U \cap U_0 = \emptyset$ imply $U \subset W$ for some $W \in \mathcal{W}(U_0, \mathcal{U})$.

We notice that the condition (1)(i) above implicitly implies that $X$ has no isolated points.

**Theorem 2.4.** *([3,8]) Let $X$ and $Y$ be uniform spaces with uniformities $\{\mathcal{U}_\alpha\colon \alpha \in A\}$ and $\{\mathcal{V}_\beta\colon \beta \in B\}$ respectively. Then $X$ and $Y$ are uniformly homeomorphic if and only if $\{\mathcal{U}_\alpha\colon \alpha \in A\}$ and $\{\mathcal{V}_\beta\colon \beta \in B\}$ are lattice-isomorphic by a correspondence preserving the orders $<$ and $\lhd$.*

*Furthermore, let $X$ and $Y$ be complete uniform spaces having the uniform bases $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ for the uniformities of $X$ and $Y$ respectively that are also sublattices of the uniformities. If $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ satisfy the condition* (1) *above, then $X$ and $Y$ are uniformly homeomorphic if and only if $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ are lattice-isomorphic.*

Nagata furthered his interest in this direction, and he characterized the uniform topology of a complete metric space by use of a lattice consisting of finite uniform coverings and a directed set of extended uniform neighborhoods. A real-valued function $f$ of a metric space $(X, d)$ is called an *extended uniform neighborhood* $f$, if $f$ satisfies the following conditions:

(i) There exists $\varepsilon > 0$ such that $f(x) \geqslant \varepsilon$ for each $x \in X$.
(ii) If $f(x) \leqslant \frac{1}{2n}$ and $d(x, y) \leqslant \frac{1}{2n}$, then $f(y) \leqslant \frac{1}{n}$.

For extended uniform neighborhoods $f$ and $g$ of $X$ we mean $f \leqslant g$ by $f(x) \leqslant g(x)$ for each $x \in X$. Let $(X, d)$ be a complete metric space. We consider the following conditions:

(a) Let $\mathcal{L}_f(X)$ denote a lattice consisting of open finite uniform coverings of $X$ satisfying the following conditions:
   (1) If $\mathcal{U}, \mathcal{V} \in \mathcal{L}_f(X)$, then $\mathcal{U} \cup \mathcal{V} \in \mathcal{L}_f(X)$.
   (2) For each pair of disjoint open sets $U, V$ of $X$ with $V \neq \emptyset$ there exists $\mathcal{W} \in \mathcal{L}_f(X)$ such that $U \in \mathcal{W}$ and $V \notin \mathcal{W}$, and
   (3) $\mathcal{L}_f(X)$ is a basis of the family of all finite uniform coverings of $X$.
(b) Let $D(X)$ be a directed set of extended uniform neighborhoods which satisfies the following conditions:
   (4) There is a subset $\{e_n\colon n = 1, 2, \ldots\}$ of $D(X)$ such that (i) for every $f \in D(X)$ there is $e_n$ such that $e_n \leqslant f$, (ii) $e_n \geqslant e_{n+1}$ for each $n$, (iii) $\lim_{n\to\infty} e_n(x) = 0$ for each $x \in X$, and (iv) for every $\varepsilon > 0$ and $x \in X$ there exists $f \in D(X)$ such that $f(x) < \varepsilon$, $f(y) \geqslant e_n(y)$ for each $y \in X$ with $d(x, y) \geqslant \frac{1}{n}$.
   (5) If $f, g \in D(X)$, then $f \vee g \in D(X)$.

**Theorem 2.5.** *([10]) Let $X$ and $Y$ be complete metric spaces having the lattices $\mathcal{L}_f(X)$ and $\mathcal{L}_f(Y)$ consisting of finite uniform coverings of $X$ and $Y$ respectively, which satisfy the condition* (a) *above. Then $X$ and $Y$ are uniformly homeomorphic if and only if $\mathcal{L}_f(X)$ and $\mathcal{L}_f(Y)$ are lattice-isomorphic.*

**Theorem 2.6.** *([14]) Let $X$ and $Y$ be complete metric spaces having the directed sets $D(X)$ and $D(Y)$ of extended uniform neighborhoods of $X$ and $Y$, respectively, which satisfy the condition* (b) *above. Then $X$ and $Y$ are uniformly homeomorphic if and only if $D(X)$ and $D(Y)$ are order-isomorphic.*

In [2], Nagata characterized a uniform structure whose completion is the Čech–Stone compactification (Wallman compactification). Let $X$ be a uniform space with a uniformity $\{\mathcal{U}_\alpha\colon \alpha \in A\}$. We say a uniform space $X$ is *Čech u-normal* (resp. *u-normal*) if for every pair of disjoint zero sets (resp. closed sets) $E$ and $F$ of $X$ there is an $\alpha \in A$ such that $\mathrm{St}(E, \mathcal{U}_\alpha) \cap F = \emptyset$.

**Theorem 2.7.** *([2]) Let $X$ be a uniform space with a uniformity $\{\mathcal{U}_\alpha\colon \alpha \in A\}$ . Then the completion of $X$ is the Čech–Stone compactification $\beta X$* (*resp. the Wallman compactification $w(X)$*) *if and only if $X$ is Čech u-normal* (*resp. u-normal*) *and totally bounded.*

In the same paper, he also studied the u-normality of metric spaces. A metric space $(X, d)$ is said to be a *u-normal* space [2] if for every pair of disjoint closed sets $E$ and $F$ of $X$, $d(E, F) > 0$. It is clear that every compact metric space is a u-normal space, and as easily seen from the theorem below, every u-normal metric space is a complete metric space. He obtained some characterizations of the u-normality of metric spaces.

**Theorem 2.8.** *([2]) Let $X$ be a metrizable space. Then the following conditions are equivalent*:

(1) *$X$ admits a u-normal metric.*
(2) *The finest uniformity of $X$ is metrizable.*
(3) *The set of all non-isolated points of $X$ is compact.*

**Theorem 2.9.** *([2]) Let $(X, d)$ be a metric space. Then the following conditions are equivalent*:

(1) *$(X, d)$ is u-normal.*
(2) *There exists a compact subset $K$ of $X$ such that for each $\varepsilon > 0$ $X - S_\varepsilon(K)$ is uniformly discrete, i.e., there is $\delta > 0$ such that $d(x, y) \geqslant \delta$ for each pair of distinct points $x, y \in X - S_\varepsilon(K)$, where $S_\varepsilon(K) = \{x \in X\colon d(x, K) < \varepsilon\}$.*
(3) *Every bounded real-valued continuous function of $X$ is uniformly continuous.*

The u-normal spaces are sometimes called *UC-spaces* or *Atsuji spaces* in the literature, and u-normal spaces are studied by several authors: Rainwater, Willard, Pettis, Beer, Himmelberg, Prikry and van Vleck, etc. The reader referred to [90] for a brief summary on the u-normal spaces.

Furthermore, Nagata showed that the uniform topology of a uniform space can be determined by the uniform convergence of uniform directed sequences of points [12], and he characterized the lattice of lower semi-continuous non-negative bounded functions on $T_1$-spaces [6].

In [18], Nagata considered the relations between continuous functions and locally finite open coverings of topological spaces as well as an extension of a continuous mapping from a closed subset of a uniform space.

**Theorem 2.10.** *([18]) Let $X$ be a fully normal uniform space with a uniformity $\{\mathcal{U}_\alpha\colon \alpha \in A\}$ and $Y$ a uniform space with a uniformity $\{\mathcal{V}_\beta\colon \beta \in B\}$ such that $|A| = |B| = \tau$. If $f : F \to Y$ is a continuous mapping from a closed subset $F$ of $X$ onto $Y$, then there are a uniform space $Z$ with a uniformity $\{\mathcal{W}_\gamma\colon \gamma \in \Gamma\}$ with $|\Gamma| = \tau$ and a continuous mapping $\tilde{f} : X \to Z$ such that $Y$ is a closed sub-uniform space of $Z$ and $\tilde{f}|_{X-F} : X - F \to Z - Y$ is a homeomorphism.*

The theorem above generalized a classical theorem of Hausdorff [Hu] (cf. [E1, Problem 4.5.20(a)]): Let $F$ be a closed subset of a metric space and $f : F \to Y$ a mapping from $F$ onto a metric space. Then there are a metric space $Z$ and a continuous mapping $\tilde{f} : X \to Z$ such that $Y$ can be isometrically embedded in $Z$ and $\tilde{f}|_{X-F} : X - F \to Z - Y$ is a homeomorphism.

## 3. Metrization and generalized metric spaces

The most well-known general metrization theorem, commonly known as the **Nagata–Smirnov Metrization Theorem**, was given independently by Jun-iti Nagata [5] and Yu.M. Smirnov [Sm] and appeared in separate papers in 1950 and 1951.

**Theorem 3.1.** *([5,Sm]) A topological space $X$ is metrizable if and only if $X$ is $T_3$ and has a $\sigma$-locally finite open base.*

Remarkably, about the same time, R.H. Bing had developed a metrization theorem using a slightly different approach for the base condition – using a $\sigma$-discrete base instead of a $\sigma$-locally finite base. In this form, the theorem is sometimes called the **Bing–Nagata–Smirnov Metrization Theorem**.

**Theorem 3.2.** *([Bi,5,Sm]) A topological space $X$ is metrizable if and only if $X$ is $T_3$ and has a $\sigma$-discrete open base.*

These elegant theorems have survived and served as a foundation of general metrization theory for over 60 years and will continue to do so. A student of metrization theory cannot help but appreciate the fundamental notion of being able to bootstrap up to a metric using only purely topological conditions on an open base – in this case, the $\sigma$-locally finite open base is a simple topological condition but yet powerful enough to do the job.

Nice covers by metrizable subsets can lead to metrizability. These are essentially due to Nagata and Smirnov. Part (b) of the next theorem would clearly follow from Part (a). Part (b) can be useful and is sometimes called the Smirnov Metrization Theorem but there are similar ideas found in both of [5] and [Sm].

**Theorem 3.3.** *([5,Sm])*

(a) *A $T_1$-space which has a locally finite covering by closed metrizable sets is itself metrizable.*
(b) *A paracompact, locally metrizable space $X$ is metrizable.*

The next metrization theorem by Nagata is of a much different nature than the Nagata–Smirnov Metrization Theorem.

**Theorem 3.4.** *([21,41,77]) A $T_1$-space $X$ is metrizable if and only if every $x \in X$ has two sequences $\{U_n(x)\}_{n=1}^{\infty}$, $\{S_n(x)\}_{n=1}^{\infty}$, of neighborhoods such that*

(i) *$\{U_n(x)\colon n \in \mathbb{N}\}$ is a neighborhood base at $x$,*
(ii) *$y \notin U_n(x)$ implies $S_n(y) \cap S_n(x) = \emptyset$,*
(iii) *$y \in S_n(x)$ implies $S_n(y) \subseteq U_n(x)$.*

It is clear that the neighborhoods $U_n(x)$ and $S_n(x)$ are simply trying to mimic the relationship between the $1/n$ balls and the $1/2n$ balls, respectively, in a metric space. In fact, given a metric space $X$, defining $U_n(x)$ and $S_n(x)$ as the $1/n$ balls and the $1/2n$ balls about $x$ will quickly give the conditions (i), (ii), and (iii). The other direction is much more difficult to prove – making it even more interesting that these simple conditions actually do give metrizability. The proof of this theorem given by Nagata in [77] (Modern General Topology, 1985) is different than the original proof given in [21]. In the later proof he shows that a $T_1$-space $X$ having the neighborhoods satisfying (i), (ii) and (iii) is a paracompact developable space. For paracompactness, Nagata takes advantage of the characterization of paracompact spaces using $\sigma$-cushioned open refinements due to E.A. Michael [Mi]. It is interesting to look at (i) and (ii) first by themselves. It is clear that (ii) gives $\overline{S_n(x)} \subseteq U_n(x)$, so along with (i), we see that $X$ is regular. A slightly deeper look shows that for any subset $A \subseteq X$ the collection of neighborhoods $\{S_n(x)\colon x \in A\}$ is cushioned in $\{U_n(x)\colon x \in A\}$. This can now be used to find a $\sigma$-cushioned open refinement of any open cover $\mathcal{U}$. To show that $X$ is developable one can define $\mathcal{W}_n = \{S_n(x)^{\circ}\colon x \in X\}$ and show that $\{\mathcal{W}_n\colon n \in \mathbb{N}\}$ is a development (using (i), (ii) and (iii)).

Nagata has shown in [24], and further discussed the idea much later in [89], that Theorem 3.4 easily implies the sufficiency of most of the previously known general metrizability conditions. Some examples are given below in Corollary 3.6.

It is also shown in [24] that condition (iii) can be dropped if the space $X$ is assumed to be Čech complete. It is worthwhile to give a separate statement of this theorem.

**Theorem 3.5.** *([24]) A Čech complete space $X$ is metrizable if and only if every $x \in X$ has two sequences $\{U_n(x)\}_{n=1}^{\infty}$, $\{S_n(x)\}_{n=1}^{\infty}$, of neighborhoods such that*

(1) *$\{U_n(x)\colon n \in \mathbb{N}\}$ is a neighborhood base at $x$,*
(2) *$y \notin U_n(x)$ implies $S_n(y) \cap S_n(x) = \emptyset$.*

For specific applications of Theorem 3.4 Nagata offers the following corollary, in [77], containing the proofs of three metrization theorems. Part (A) is due to A.H. Frink [Fr] and (B), (C) are due to K. Morita [Mo1]. See [77] and [89] for more applications of Theorem 3.4.

**Corollary 3.6.** *([77,Fr,Mo1]) A $T_1$-space $X$ is metrizable if and only if it satisfies one of the following conditions*:

(A) *There exists a neighborhood basis $\{W_n(p)\}_{n=1}^{\infty}$ at each $p \in X$ such that for every $n$ and $p \in X$, there exists $m = m(n, p)$ for which $W_m(p) \cap W_m(q) \neq \emptyset$ implies $W_m(q) \subseteq W_m(p)$.*
(B) *There exists a sequence $\{\mathcal{F}_n\}_{n=1}^{\infty}$ of closure-preserving closed coverings of $X$ such that for every neighborhood $U(p)$ of any point $p \in X$, there is an $n$ for which $\mathrm{St}(p, \mathcal{F}_n) \subseteq U(p)$.*
(C) *There exists a sequence $\{\mathcal{U}_n\}_{n=1}^{\infty}$ of open coverings of $X$ such that $\{\mathrm{St}^n(p, \mathcal{U}_n)\colon n \in \mathbb{N}\}$ is a local basis of each point $p$ of $X$.*

An interesting version of Theorem 3.4 is given in [23] where the complexity of the two sequences $\{U_n(x)\}_{n=1}^{\infty}$ and $\{S_n(x)\}_{n=1}^{\infty}$ is traded for the simplicity of one local base $\{V_n(p)\colon n \in \mathbb{N}\}$ using more complex conditions on the indexing.

**Theorem 3.7.** *([23]) A $T_1$-space is metrizable if and only if each $p \in X$ has a neighborhood basis $\{V_n(p)\colon n \in \mathbb{N}\}$ such that*

(a) *for every $p \in X$ and $n \in \mathbb{N}$ there exists $m = \alpha(p, n)$ such that $p \in V_m(q)$ implies $V_m(q) \subseteq V_n(p)$, and*
(b) *for every $p \in X$ and $n$ there exists $l = \beta(p, n)$ such that $q \in V_l(p)$ implies $p \in V_n(q)$.*

The proof of the following intriguing corollary is left to the interested reader.

**Corollary 3.8.** *([23]) A $T_1$-space $X$ is metrizable if and only if one can assign to each $p \in X$ a neighborhood basis $\{V_n(p)\colon n \in \mathbb{N}\}$ such that for every $A \subseteq X$, $\bigcap_{n=1}^{\infty} \bigcup_{p \in A} V_n(p) = \overline{A}$.*

In [5], where the proof of Theorem 3.1 was given, Nagata also had a metrizability theorem using a family of continuous real-valued functions in the spirit of a "partition of unity." An extension of these ideas is given in [15] and [18] in order to characterize paracompactness and metrizability in terms of a family $\{f_\alpha\colon \alpha \in \lambda\}$ of continuous, real-valued functions on $X$. The index set $\lambda$ below can be considered as a cardinal number, so that every subset $B$ of $\lambda$ has a natural well-ordering.

**Theorem 3.9.** *([5,15,18]) Suppose $\{f_\alpha\colon \alpha \in \lambda\}$ denotes a family of continuous, real-valued functions on a Hausdorff space $X$. The following are true:*

(1) *If the sets $V_\alpha = \{x \in X\colon f_\alpha(x) > 0\}$ cover $X$ and if $\sup_{\beta<\alpha} f_\beta$ is continuous for all $\alpha \in \lambda$, then the cover $\{V_\alpha\colon \alpha \in \lambda\}$ has a locally finite open refinement.*
(2) *$X$ is paracompact if and only if for every open covering $\{V_\alpha\colon \alpha \in \lambda\}$ there exists a family $\{f_\alpha\colon \alpha \in \lambda\}$ such that $f_\alpha(X \setminus V_\alpha) = 0$, $\sup_{\alpha<\lambda} f_\alpha = 1$, and $\sup_{\beta \in B} f_\beta$ is continuous for every $B \subseteq \lambda$.*
(3) *$X$ is metrizable if and only if there exists a family $\{f_\alpha\colon \alpha \in \lambda\}$ such that $\sup_{\beta \in B} f_\beta$ and $\inf_{\beta \in B} f_\beta$ are continuous for every $B \subseteq \lambda$, and such that for every $x \in X$ and neighborhood $U$ of $x$ there exists an $f_\alpha$ with $f_\alpha(x) < \varepsilon$ and $f_\alpha(X \setminus U) \geqslant \varepsilon$ for some $\varepsilon > 0$.*

Starting in the early sixties and continuing for 20 or 30 years the topology community saw the introduction of many interesting generalized metric spaces which were studied on their own merits and which could be used to factor metrizability (and developability) into component parts. This idea was fully embraced by the school of Japanese topologists and the contributions by Jun-iti Nagata and other Japanese is especially noteworthy.

The notions of spaces with a $\sigma$-locally finite network and of spaces with a $\sigma$-closure preserving network were introduced by A. Okuyama in [Ok2]. (The term "net" was used; currently it is more common to use the term "network.") In [Ok3] Okuyama defined the class of $\sigma$-spaces to be the spaces with a $\sigma$-locally finite network. At that time it was still not known whether the $\sigma$-spaces were any different than the spaces with a $\sigma$-closure preserving network. This was clarified by F. Siwiec and J. Nagata in [42] with the following result.

**Theorem 3.10.** *([40]) For a $T_3$-space $X$ the following conditions are equivalent:*

(1) *$X$ has a $\sigma$-closure-preserving network,*
(2) *$X$ is a $\sigma$-space (has a $\sigma$-locally finite network),*
(3) *$X$ has a $\sigma$-discrete network.*

Not only did this theorem show that the existence of a $\sigma$-locally finite network is equivalent to the existence of a $\sigma$-closure-preserving network, it also immediately shows that the class of $\sigma$-spaces is preserved under the action of closed (continuous) mappings.

Siwiec and Nagata also gave several factorization theorems characterizing metrizability by using two or three generalized metric notions. A sampling is given the next theorem.

**Theorem 3.11.** *([40]) For a $T_3$ space $X$ the following conditions are equivalent:*

(1) *$X$ is metrizable,*
(2) *$X$ is a $\sigma^{\#}$-space and an $M$-space,*
(3) *$X$ is a collectionwise normal $\sigma$-space which is also a $w\Delta$-space.*

For several definitions below we say that a sequence of covers $\{\mathcal{S}_n\}_{n=1}^{\infty}$ of a topological space $X$ satisfies $(*)$ or $(**)$, accordingly if one of the following is true:

$(*)$ for any $x \in X$ and $x_n \in \mathrm{St}(x, \mathcal{S}_n)$, all $n \in \mathbb{N}$, the sequence $\{x_n\}_{n=1}^{\infty}$ has a cluster point.

$(**)$ for any $x \in X$ and $x_n \in \mathrm{St}^2(x, \mathcal{S}_n)$, all $n \in \mathbb{N}$, the sequence $\{x_n\}_{n=1}^{\infty}$ has a cluster point.

A space $X$ is said to be an *M-space* [Mo3] if and only if there exists a normal sequence $\{\mathcal{G}_n\}_{n=1}^{\infty}$ of open covers of $X$ satisfying: $(*)$.

A space $X$ is said to be an *$M^*$-space* (*$M^{\#}$-space*) [Is1] if and only if there exists a sequence $\{\mathcal{F}_n\}_{n=1}^{\infty}$ of locally-finite (closure-preserving) closed covers of $X$ satisfying: $(*)$.

A space $X$ is said to be an *wM-space* [Is2] if and only if there exists a sequence $\{\mathcal{G}_n\}_{n=1}^{\infty}$ of open covers of $X$ satisfying: $(**)$.

A space $X$ is said to be an *$w\Delta$-space* [Bo] if and only if there exists a sequence $\{\mathcal{G}_n\}_{n=1}^{\infty}$ of open covers of $X$ satisfying: $(*)$.

A space $X$ is a (strong) $\Sigma$-space [Na1] if and only if there is a $\sigma$-locally finite collection $\mathcal{F}$ of closed subsets of $X$ and a cover $\mathcal{C}$ by countably-compact (compact) sets such that if $C \in \mathcal{C}$ and $C \subseteq U$, where $U$ is open, then $C \subseteq F \subseteq U$ for some $F \in \mathcal{F}$.

These "generalized metric" spaces were quickly found to be useful in metrization theory, products of normal and paracompact spaces and the study of preservation and characterization under mappings and inverse mappings. Our intention here is not to give a comprehensive survey of these and other properties (some yet to follow) but to bring up a few basic connections between these and others in order to highlight some of Nagata's important work. There are several surveys available, including several by Nagata. In particular we mention his early 1971 article [53] and the excellent survey by G. Gruenhage [Gr2] in 1984.

**Theorem 3.12.** *([Mo3]) A space $X$ is an $M$-space if and only if there is a quasi-perfect map $f : X \to Y$ onto a metric space $Y$.*

**Theorem 3.13.** *([Is1,Is2]) Every $M$-space is an $M^*$-space. Every normal $M^*$-space is an $M$-space.*

There are no direct implications (without additional conditions) between the $M$-spaces and the $p$-spaces [A1] of Arhangel'skii but the following theorem gives a very useful and informative connection.

**Theorem 3.14.** *([Mo3,A2]) The following are equivalent conditions on a space $X$:*

(1) *$X$ is a paracompact $M$-space.*
(2) *$X$ is a paracompact $p$-space.*
(3) *$X$ can be mapped onto some metric space $Y$ by a perfect mapping.*

It is clear from Theorems 3.12 and 3.14 that if $X$ is an $M$-space in which every closed countably compact subset is compact then $X$ is actually a paracompact $p$-space.

Nagata uses this theorem and work in [45] to characterize paracompact $M$-spaces in the following way:

**Theorem 3.15.** *([45,46]) A space $X$ is a paracompact $M$-space if and only if $X$ is homeomorphic to a closed subset of a product of a metric space and a compact $T_2$-space.*

Using Theorem 3.12 it is easy to show that every closed subset of the product of a metric space $Y$ and a countably compact space $Z$ is an $M$-space. Nagata was aware of this and because of Theorem 3.15 he asked the following question (converse of the previous remark) which he posed in [45] and [59].

**Problem.** Is every $M$-space homeomorphic to a closed subset of a product of a metric space and a countably compact space?

Prior to [59], K. Morita had introduced the notion of a countably-compactification of a space $X$ and this appeared in [Mo4]. A space $S$ is a countably-compactification of $X$ if $S$ is countably compact, contains $X$ as a dense subspace, and every closed countably compact subset of $X$ remains closed in $S$. Morita and Nagata realized that the problem above is equivalent to the question: *Is every M-space countably-compactifiable?*

The problem was answered negatively by D. Burke and E.K. van Douwen in [BuvD] and independently by A. Kato in [Ka] where examples were given of locally compact $M$-spaces which were not countably-compactifiable. It is interesting and worthwhile to notice that there has been some very recent results on "Nagata's conjecture" which are very set-theoretic in nature. L. Soukup has shown in [So] that there is a c.c.c poset $P$ of size $2^{\omega}$ such that for any first-countable regular space $X$ in the ground model $V$, which is an $M$-space in $V^P$, is a closed subspace of the product of a countably compact space and a metric space in $V^P$. Soukup actually shows that any first-countable regular space $X$ in the ground model $V$ has a first countable, countably compact extension in $V^P$.

In the spirit of asking which spaces can be expressed as a "nice" continuous image of a "nice" space, Nagata offers the following characterizations of certain images of $M$-spaces.

**Theorem 3.16.** *([45]) A regular space $Y$ is a $q$-space (in the sense of E. Michael) if and only if there is an $M$-space $X$ and a continuous open surjection $f : X \to Y$.*

**Theorem 3.17.** *([43]) A regular space $Y$ is a quasi-$k$-space if and only if there is an $M$-space $X$ and a quotient map $f : X \to Y$.*

**Theorem 3.18.** *([43]) A space $Y$ is a $k$-space if and only if there is an paracompact $M$-space $X$ and a quotient map $f : X \to Y$.*

**Theorem 3.19.** *([43]) A space $Y$ is a bi-$k$-space if and only if there is an $M$-space $X$ and a bi-quotient map $f : X \to Y$.*

Theorem 3.15 leads Nagata to ask about other classes of generalized metric spaces (of weight $\alpha$) which can be embedded as a closed subspace of some "nice" universal space or which is the image of a "nice" space under a continuous map. We follow with a selection of such characterizations by Nagata.

In the following, where $|A|$ denotes the cardinality of a set $A$, $D(A)$ denotes the Cantor discontinuum $\prod_{\alpha\in A}\{0,1\}$ and $N(A)$ the Baire 0-dimensional metric space $\prod_{n\in\omega} A$ (with the discrete topology on $A$). Furthermore, $H(A)$ will denote the generalized Hilbert space and $P(A)$ denotes the product $\prod_{\alpha\in A} I_\alpha$ (where $I_\alpha$ is a copy of the unit interval $[0,1]$).

**Theorem 3.20.** *([45,46]) Every paracompact $M$-space $Y$ of weight $w(Y) = |A|$ is the image of a 0-dimensional, paracompact $M$-space $X$ by a perfect map $f : X \to Y$, where $X$ is a closed subset of the product of $D(A)$ and a subset of $N(A)$.*

**Theorem 3.21.** *([45,46]) A space $X$ with weight $|A|$ is a paracompact $M$-space if and only if $X$ is homeomorphic to a closed subset of $S \times P(A)$, where $S$ is a subset of $H(A)$.*

**Theorem 3.22.** *([45,46]) Every paracompact Čech complete space $Y$, with weight $|A|$, is the image of a closed subset of $D(A) \times N(A)$ by a perfect map.*

**Theorem 3.23.** *([45,46]) A space $X$ with weight $|A|$ is a paracompact, Čech complete space if and only if $X$ is homeomorphic to a closed subset of $H(A) \times P(A)$.*

The spaces that can be expressed as the image of a metric space under a closed continuous mapping are called *Lašnev spaces*. N. Lašnev [La] proved the following decomposition theorem.

**Theorem 3.24.** *([La]) If $X$ is a metric space and $f : X \to Y$ is a continuous closed surjection then $Y$ has the following form:*

(a) *$Y = Y_0 \cup (\bigcup_{n=1}^\infty Y_n)$, where $f^{-1}(y)$ is compact for each $y \in Y_0$ and, for all $n \in \mathbb{N}$, $Y_n$ is a closed discrete subspace of $Y$.*
(b) *Moreover the subspace $Y_0$ is metrizable and, if the metric space $X$ has weight $w(X) = \tau$, then the $\sigma$-discrete part has cardinality $|\bigcup_{n=1}^\infty Y_n| \leqslant \tau$.*

Earlier, Morita [Mo2] had proved a similar decomposition theorem for closed images of paracompact locally-compact spaces (without the metrizable condition on $Y_0$) and since the appearance of Theorem 3.24(a) several authors have strengthened the theorem by weakening the conditions on the domain space. Nagata was interested in unifying some of these decomposition theorems by finding a general condition on the domain space which would be implied by some of the other conditions. He was partly successful in [50] where he introduced two classes of spaces called QSM-spaces and SSM-spaces. We give these somewhat technical definitions below – the interested reader may find a way to extend the results.

A topological space $X$ is called a *QSM-space* if it has a cover $\{F_\alpha\colon \alpha \in \Lambda\}$ by countably compact closed sets (called *kernels*) and if each $F_\alpha$ has a neighborhood base $\{U_{\alpha n}\colon n \in \mathbb{N}\}$ such that, for each $n \in \mathbb{N}$, $\{F_\alpha\colon \alpha \in \Lambda\}$ is cushioned in $\{U_{\alpha n}\colon \alpha \in \Lambda\}$.

It is easy to verify that all semi-metric spaces and all $M$-spaces are QSM. Nagata offers the following decomposition theorem using the notion of a QSM-space.

**Theorem 3.25.** *([50,51]) Let $X$ be a normal QSM space and let $f : X \to Y$ be a closed continuous map onto a space $Y$. Then, $Y = Y_0 \cup (\bigcup_{n=1}^\infty Y_n)$ where every $Y_n$, for $n > 1$, is a closed discrete subspace of $Y$ and for each $y \in Y_0$, $f^{-1}(y)$ is $k$-countably compact (i.e., every sequence $\{F_{\alpha(i)}\}_{i=1}^\infty$ of kernels with each $F_{\alpha(i)} \cap f^{-1}(y) \neq \emptyset$ has a cluster point).*

A topological space $X$ is called an *SSM-space* if each element $x \in X$ has a sequence $\{U_n(x)\}_{n=1}^\infty$ of neighborhoods (not necessarily open) such that

(i) $y \in U_n(x)$ implies $x \in U_n(y)$,
(ii) if $\{x_i\colon i \in \mathbb{N}\}$ is a discrete set in $X$ there are two strictly increasing sequences $\{n(j)\}_{j=1}^\infty$ and $\{i(j)\}_{j=1}^\infty$ in $\mathbb{N}$ such that $\{U_{n(j)}(x_{i(j)})\}$ is a closed discrete collection in $X$.

It can be observed that the SSM spaces include normal semi-metric spaces, $wM$-spaces, $M$-spaces, $M^*$-spaces, normal first-countable $\sigma$-spaces, first-countable stratifiable spaces and others (no attempt here to list mutually exclusive examples).

For the SSM-spaces Nagata has the following decomposition theorem.

**Theorem 3.26.** *([50,51]) Let $X$ be SSM and let $f : X \to Y$ be a closed continuous map onto a space $Y$. Then $Y = Y_0 \cup (\bigcup_{n=1}^{\infty} Y_n)$ where every $Y_n$, for $n > 1$, is a closed discrete subspace of $Y$ and for each $y \in Y_0$, $f^{-1}(y)$ is countably compact.*

One difference in each these decompositions (as opposed to the original Lašnev decomposition) is a countably compact condition on the $f^{-1}(y)$ instead of a compact condition. If the space $X$ is isocompact (closed countably compact subsets are compact) then the $f^{-1}(y)$ turn out to be compact – so in some cases this is not really a weakening of the decomposition.

It is worth noting that J. Suzuki [Su] proved a decomposition theorem with the result of Theorem 3.26 using a normal *pre-$\sigma$-space* (quasi-perfect preimage of a $\sigma$-space) as the domain space. The class of pre-$\sigma$-spaces includes the class of $\sigma$-spaces and $M$-spaces.

In addition to the general study of characterizing classes of topological spaces by images or preimages by continuous (single-valued) mappings Nagata has made significant contributions by taking advantage of the use of multi-valued maps. Much of the terminology for multi-valued maps used here can be discerned by comparing with terminology of single-valued mappings. However, for convenience of the reader, we define some of the technical terms relating to the theorems on multi-valued maps. We will refer the interested reader to [47], [51], or [54] in case there are other questions about terminology or notation.

Let $f$ be a multi-valued mapping from a space $X$ to a space $Y$ (assuming $f(x) \neq \emptyset$ for every $x \in X$ and $f^{-1}(y) \neq \emptyset$ for every $y \in Y$). If, for every $y \in Y$ there is $x \in f^{-1}(y)$ such that $f^{-1}\langle V\rangle = \{x \in X\colon\ f(x) \subset V\}$ is a neighborhood of $x$ for every neighborhood $V$ of $y$ then $f$ is called *selection continuous*. If, for every $y \in Y$ and neighborhood $V$ of $y$ there is $x \in f^{-1}(y)$ such that $f^{-1}\langle V\rangle$ is a neighborhood of $x$ then $f$ is called *w. selection continuous*. The map $f$ is called a *s-perfect map* if $f$ is closed, selection continuous and $f^{-1}(y)$ is compact for every $y \in Y$. Also, $f$ is *$Y$-countably compact* if $f(x)$ is countably compact for every $x \in X$. In case $X$ has a metric $d$, $f$ is said to be a *$\pi$-map* if for any $y \in Y$ and closed $G \subseteq Y$, $d(f^{-1}(y), f^{-1}(G)) > 0$.

**Proposition 3.27.** *([47]) Let $X$ be a regular $\sigma$-space and $Y$ a topological space. If there is a closed, w. selection continuous map $f$ from $X$ to $Y$, then $Y$ is a $\sigma$-space.*

Proposition 3.27 can be used to prove the sufficiency in the next theorem. The necessity is quite a bit more involved and difficult.

**Theorem 3.28.** *([47,54]) A space $Y$ is a $\sigma$-space if and only if there is a subspace $X$ of the Baire zero-dimensional metric space $N(A)$ (of weight $|A|$) and a s-perfect multi-valued map from $X$ to $Y$.*

**Proposition 3.29.** *([47]) Let $f$ be a quasi-perfect, $Y$-countably compact multi-valued map from $X$ to $Y$. If $X$ is an $M^*$-space then $Y$ is also an $M^*$-space.*

Nagata remarks that the above proposition generalizes two theorems from [Is1] at the same time. An advantage of multi-valued maps is to provide the possibility of unifying two types of theories of single-valued maps – the theory of images and the theory of preimages.

Proposition 3.29 gives the sufficiency direction in the next theorem.

**Theorem 3.30.** *([47,54]) A space $Y$ is an $M^*$-space if and only if there is a subspace $X$ of the Baire zero-dimensional metric space $N(A)$ (of weight $|A|$) and a perfect $Y$-countably compact multi-valued map from $X$ to $Y$.*

**Theorem 3.31.** *([54]) A Tychonoff space $Y$ is a strict $p$-space if and only if there is a subspace $X$ of the Baire zero-dimensional metric space $N(A)$ (of weight $|A|$) and a multi-valued map $f$ from $X$ to $Y$, where $f$ is open, $\pi$, bi-$Y$-compact and $w$-closed.*

The $M_2$-space referred to in the next result is the $M_2$-space in the $M_1, M_2, M_3$ sequence reviewed a little later in this paper.

**Theorem 3.32.** *([54]) A regular space $Y$ is an $M_2$-space if and only if there is a subspace $X$ of Baire zero-dimensional metric space $N(A)$ and a multi-valued map $f$ from $X$ to $Y$, where $f$ is $q$-open and $q$-closed.*

As an application to Theorem 3.32 and its proof, Nagata offers the following characterization of first-countable $M_2$-spaces.

**Theorem 3.33.** *([54]) A regular space $Y$ is a first-countable $M_2$ space if and only if there is a subspace $X$ of Baire zero-dimensional metric space $N(A)$ and a continuous single-valued map $f$ from $X$ onto $Y$ which is almost-open and $q$-closed.*

In [54] Nagata also characterizes stratifiable spaces (and semi-stratifiable spaces) as images of subspaces of $N(A)$ under single-valued maps that he calls stratifiable maps (and semi-stratifiable maps). These maps are weaker than being closed maps. See [54] for details.

In the next set of theorems, using half-metric spaces and half-$M$-spaces as the nice domains for giving images under continuous (single-valued) functions, Nagata uses the multi-valued constructions given above to assist in the proofs. These constructions can be quite involved – see [52] and [54] for details.

**Definition.** Let $(X, X')$ be a pair where $X'$ is a subspace of $X$. The pair $(X, X')$ is called a *half-metric space*, with the *metric part* $X'$, if $X$ has a sequence $\{\mathcal{U}_n\}_{n=1}^{\infty}$ of locally-finite open covers satisfying condition (H).

(H) For every $x \in X'$ and every neighborhood $V$ of $x$ in $X$ there is $U \in \bigcup_{n=1}^{\infty} \mathcal{U}_n$ for which $x \in U \subseteq V$.

**Theorem 3.34.** *([52,54]) The following conditions are equivalent for a regular space $Y$:*

(1) *$Y$ is a $\sigma$-space.*
(2) *There is a half-metric space $(X, X')$ and a perfect map $f : X \to Y$ satisfying $f(X') = Y$.*
(3) *There is a half-metric space $(X, X')$ and a closed continuous map $f : X \to Y$ satisfying $f(X') = Y$.*
(4) *There is a compact space $C$ and a half-metric space $(X, X')$ such that $X$ is a closed subset of the product $C \times Y$ and $\pi_Y(X') = Y$.*

**Definition.** Let $(X, X')$ be a pair where $X'$ is a subspace of $X$. The pair $(X, X')$ is called a *half-$M$-space* if $X$ has a **normal** sequence $\{\mathcal{U}_n\}_{n=1}^{\infty}$ of open covers satisfying condition (M).

(M) If $x \in X'$ and $x_n \in \mathrm{St}(x, \mathcal{U}_n)$, for every $n \in \mathbb{N}$, the sequence $\{x_n\}_{n=1}^{\infty}$ has a cluster point in $X$.

**Theorem 3.35.** *([52]) A space $Y$ is a $\Sigma$-space if and only if there is a half-$M$-space $(X, X')$ and a perfect map $f : X \to Y$ such that $f(X') = Y$.*

**Theorem 3.36.** *([51,54]) The following conditions are equivalent for a space $Y$:*

(1) *$Y$ is an $M^*$-space.*
(2) *There is a compact space $C$ and a closed subset $X$ of the product space $Y \times C$ where $X$ is an $M$-space as a subspace and satisfies $\pi_Y(X) = Y$.*
(3) *There is a perfect map from an $M$-space $X$ onto $Y$.*

Nagata's Theorem 3.15 characterizes paracompact $M$-spaces as the closed subsets of products of a metric space and a compact space. This motivated him to look at the $G_\delta$ subsets in the product of a metric space and a compact space. While he was not successful at first in finding an internal characterization of such spaces there are some interesting results and questions.

**Theorem 3.37.** *([55]) An $M$-space $X$ is homeomorphic to a $G_\delta$-set in the product of a metric space and compact Hausdorff space if and only if $X$ is a $p$-space.*

Nagata notices that in the proof of Theorem 3.15 that the $G_\delta$-set in the proof of Theorem 3.37 is a closed subset of $\beta X \times Y$ if $X$ is a paracompact $M$-space. Hence this corollary follows.

**Corollary 3.38.** *([55]) Every paracompact $M$-space $X$ is homeomorphic to a closed $G_\delta$-set in the product of a metric space and compact Hausdorff space.*

We recall that a topological space $X$ is called a *$G_\delta$-space* [55] if $X$ is homeomorphic to a $G_\delta$-set in the product of a metric space and a compact $T_2$ space.

Observe that every metric space and every Čech complete space is a $G_\delta$-space. Also, every $G_\delta$-space is a $p$-space since every $G_\delta$-subset of a $p$-space is a $p$-space.

Nagata poses four problems in [55]:

1. Is very $p$-space a $G_\delta$-space?
2. Give an internal characterization of $G_\delta$-spaces. (Nagata actually did give an internal characterization in the sequel [56].)

3. Is every space $X$ which is both an $M$-space and a $p$-space homeomorphic to a $G_\delta$-set $S$ in the product of a metric space and a compact Hausdorff space such that $S$ is the intersection of countably many open $F_\sigma$-sets?
4. Characterize the compact open images of $G_\delta$-spaces.

Problem 4 was probably motivated by the next theorem which effectively does give the characterization in the case when the image space is known to be a metacompact space.

**Theorem 3.39.** *([55]) A metacompact Tychonoff space $Y$ is a $p$-space if and only if there is a $G_\delta$-space $X$ and a compact open surjection $f : X \to Y$.*

Nagata gives two internal characterizations of $G_\delta$-spaces in [56] and indicates that the proofs of that and a previous theorem will give the next result which will finish our discussion of the topic on $G_\delta$-spaces.

**Theorem 3.40.** *([56]) A space $X$ is a $G_\delta$-space if and only if $X$ is homeomorphic to a closed set in the product of a metric space $M$ and a Čech complete space $Y$.*

In 1961, J. Ceder introduced three classes of generalized metric spaces denoted by $M_1$, $M_2$, and $M_3$-spaces. Each was given with a natural weakening of the metrizability condition found in the Nagata–Smirnov Metrization Theorem 3.1.

**Definition.** A regular space $X$ is an $M_1$-*space* if $X$ has a $\sigma$-closure preserving open base; $X$ is an $M_2$-*space* if $X$ has a $\sigma$-closure preserving quasi-base; $X$ is an $M_3$-*space* if $X$ has a $\sigma$-cushioned pairbase.

It is clear from the definitions that "$M_1 \Rightarrow M_2 \Rightarrow M_3$." No implications in the other directions were known until 1976 when G. Gruenhage and H. Junnila independently showed that $M_2$ and $M_3$ were equivalent. (This is a difficult result – see [Gr1] or [Gr2] for details.) It is still not known whether "$M_2 \Rightarrow M_1$" without additional conditions. $M_3$-spaces are now commonly called stratifiable spaces. To put things into the present context with other generalized metric spaces we note that stratifiable spaces are paracompact $\sigma$-spaces. (The paracompact condition is easy – the fact that stratifiable spaces are $\sigma$-spaces was shown by R.W. Heath. A proof can be found in [Gr2].) On the other end of the implications, F. Slaughter showed in 1973 that Lašnev spaces are $M_1$-spaces. (See [Gr2] for a proof.)

Now, it is worthwhile to look back at Nagata's Metrization Theorem 3.4, but specifically at conditions (i) and (ii) on the sequences $\{U_n(x)\}_{n=1}^\infty$ and $\{S_n(x)\}_{n=1}^\infty$ (which we repeat here):

(i) $\{U_n(x)\colon n \in \mathbb{N}\}$ is a neighborhood base at $x \in X$,
(ii) $y \notin U_n(x)$ implies $S_n(y) \cap S_n(x) = \emptyset$.

Looking back at the comments following the statement of Theorem 3.4 it is clear that these conditions give that $\mathcal{P} = \bigcup_{n=1}^\infty \{(S_n(x)^\circ, U_n(x))\colon x \in X\}$ is a $\sigma$-cushioned pairbase for $X$ and so $X$ is a first-countable stratifiable space. Ceder [Ce] notices this and also shows the converse; i.e., any first-countable stratifiable space has sequences $\{U_n(x)\}_{n=1}^\infty$ and $\{S_n(x)\}_{n=1}^\infty$ of neighborhoods of $x$ satisfying (i) and (ii). Spaces having such a pair $\langle\{U_n(x)\}_{n=1}^\infty, \{S_n(x)\}_{n=1}^\infty\rangle$ of sequences satisfying (i) and (ii) were called *Nagata spaces* by Ceder and the pair $\langle\{U_n(x)\}_{n=1}^\infty, \{S_n(x)\}_{n=1}^\infty\rangle$ is called a *Nagata structure*. Ceder's theorem now says:

**Theorem 3.41.** *([Ce]) A topological space $X$ is a Nagata space if and only if $X$ is first countable and stratifiable.*

The class of Nagata spaces was essentially introduced by Nagata in [24] (without the name, of course) where he was interested in testing the strength of conditions (i) and (ii) from Theorem 3.4 without (iii). Nagata gave an example of a non-metrizable separable Nagata space (later shown to be an $M_1$-space). Nagata spaces have taken on a life of their own (not originally planned) as a useful class of generalized metric spaces. Theorem 3.5 can now be restated as: A space $X$ is completely metrizable if and only if $X$ is a Čech complete Nagata space. It is now known that Nagata spaces are $M_1$ [It]. Nagata spaces are well-behaved (preserved) under subspaces, countable products and closed continuous images. Nagata spaces are paracompact $\sigma$-spaces. If there is any "fault" it may be that Nagata spaces (for being generalized metric spaces) are too close to being metrizable but, directly or indirectly, they have been the source of many results and problems for over 50 years.

The class of Nagata spaces and its roots in [24] may also be considered as the major source of motivation for studying topological spaces and various general topological properties by means of "$g$-functions" or neighborhood assignments of the form $\{V_n(x)\colon n \in \mathbb{N}\}$, for $x \in X$. A sample, proved by Nagata, would be:

**Theorem 3.42.** *([80]) A $T_0$-space $X$ is metrizable if and only if for each $x \in X$, there is an open neighborhood assignment $\{V_n(x)\colon n \in \mathbb{N}\}$ satisfying*

(1) *if* $x \in V_n(y_n)$ *and* $y_n \in V_n(x_n)$, *for all* $n \in \mathbb{N}$, *then* $x_n \to x$ ($\sigma$*-space*);
(2) *for all* $n \in \mathbb{N}$ *and all* $Y \subseteq X$, $\overline{Y} \subseteq \bigcup\{V_n(y)\colon\ y \in Y\}$.

Condition (1) in the above theorem, characterizing $\sigma$-spaces, originally comes from [HH] where authors (R.W. Heath and R. Hodel) consider $\sigma$-spaces, $w\delta$-spaces with a $G_\delta^*$-diagonal, MN-spaces, Nagata spaces and Moore spaces. The diversity of results and numerous characterizations using the neighborhood assignment approach are simply much too varied to attempt to summarize here. The interested reader may wish to look at a few papers by Nagata [75,78,80,95] as well as other authors [Ho1,Ho2,HH,Hu,JZ] for a good selection of results and problems.

Nagata gave the following theorem as a generalization of a result by V.V. Filipov who proved that a paracompact $p$-space with a point-countable base is metrizable.

**Theorem 3.43.** *([44]) A space* $X$ *is metrizable if and only if* $X$ *is a paracompact* $M$*-space with a point-countable collection* $\mathcal{U}$ *of open sets such that for any distinct* $x, y \in X$ *there exists* $U \in \mathcal{U}$ *with* $x \in U$ *and* $y \notin U$.

In [51] Nagata introduces the term "$p$-base" to denote an open collection of sets which is "$T_1$ point-separating." Nagata noticed a point-countable $p$-base condition was a simultaneous generalization of a point-countable base condition and of a condition possible using paracompact spaces with a $G_\delta$-diagonal. So, Theorem 3.43 generalizes both Filipov's theorem and a theorem by A. Okuyama [Ok1] that a paracompact $M$-space with a $G_\delta$ diagonal is metrizable. That is, it follows that a Hausdorff $M$-space with a point-countable $p$-base is metrizable. The idea of a point-countable $p$-base has since been used by Nagata and other authors as a hypothesis in further results about generalized metric spaces.

Balogh and Gruenhage [BG] and Zique and Junnila [ZJ] independently gave a negative answer to a question posed by Nagata in [95] about whether every metric space admits a metric $d$ where, for every $\varepsilon > 0$, the collection $\mathcal{B}_d(\varepsilon) = \{B_d(x, \varepsilon)\}$ of open balls is locally finite. Nagata had previously shown in [67] that every metric space admits a metric $d$ where the collection $\mathcal{B}_d(\varepsilon)$ is always closure-preserving. Nagata must have liked their solution in which they completely characterize the class of metric spaces which admit such a metric. Below, $\kappa^\omega$ denotes the Baire zero-dimensional metric space of weight $\kappa$ and $I = [0, 1]$ is the unit interval.

**Theorem 3.44.** *([BG]) The class* $\mathcal{N}$*, of metric spaces which admit a metric d where the collection* $\mathcal{B}_d(\varepsilon)$ *of open balls is locally finite, is precisely the class of spaces which are embeddable in* $\kappa^\omega \times I^\omega$*, for some cardinal* $\kappa$.

Balogh and Gruenhage now point out that a negative answer to Nagata's question is given by using any space with a non-separable component, such as a hedgehog with uncountably many spines. Such a space is not embeddable in any $\kappa^\omega \times I^\omega$.

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